Time series outlier detection using penalised regression

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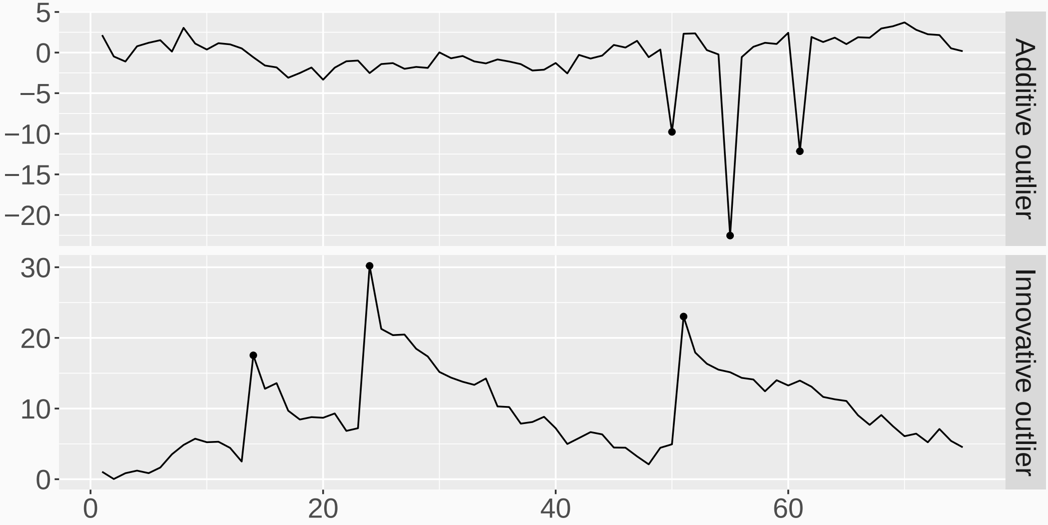
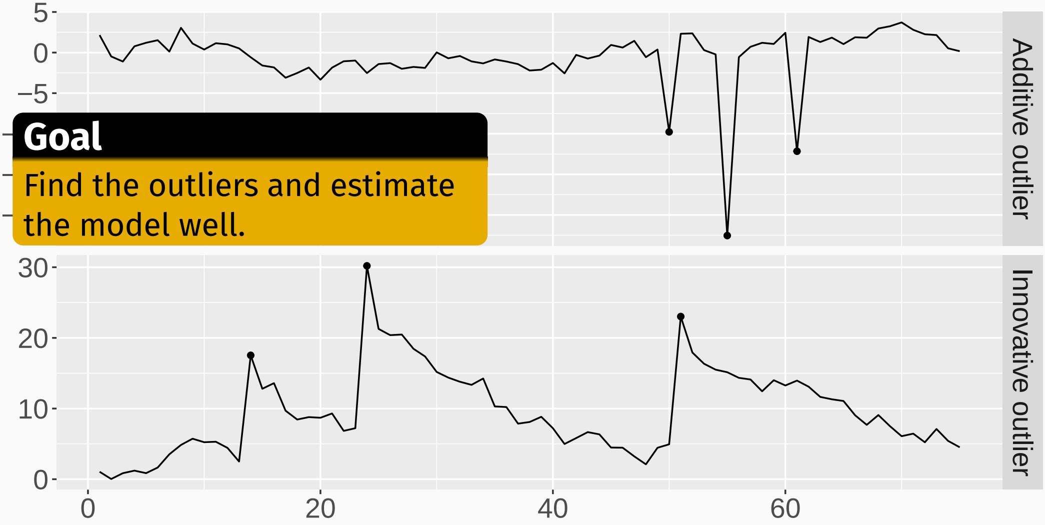
Outliers in time series



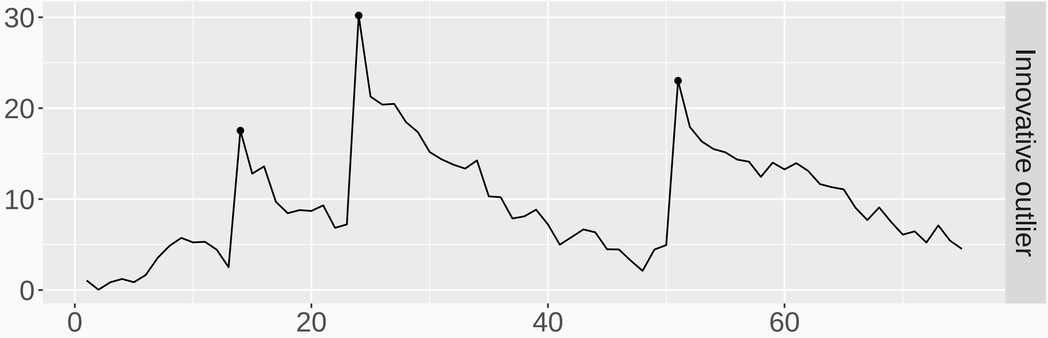
Outliers in time series



Goal

Find the outliers and estimate the model well.

Innovative outlier



Innovative outlier

$$y_t = \gamma_t + \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + \varepsilon_t$$

IPOD for Innovative Outliers

Thresholding based Iterative Procedure for Outlier Detection (She and Owen 2011)

Innovative outliers

$$y_t = \gamma_t + \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + \varepsilon_t$$

IPOD Objective function

$$\min_{\gamma} f_{\text{soft}}(\varphi, \gamma; \lambda) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{Y}\varphi - \gamma||_2^2 + \sum_{t=1}^{T} \lambda_t |\gamma_t|.$$

IPOD for Innovative Outliers

Objective function

$$\min_{\gamma} f_{\text{soft}}(\varphi, \gamma; \boldsymbol{\lambda}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{Y}\varphi - \gamma||_2^2 + \sum_{t=1}^{T} \lambda_t |\gamma_t|$$

$$\boldsymbol{y} = [y_1, y_2, \ldots, y_T]' \qquad \boldsymbol{Y} = \begin{bmatrix} y_0 & y_{-1} & \cdots & y_{-p+1} \\ y_1 & y_0 & \cdots & y_{-p+2} \\ \vdots & \ddots & & \vdots \\ y_{T-1} & y_{T-2} & \cdots & y_{T-p} \end{bmatrix}$$

IPOD for Innovative Outliers

Objective function

$$\min_{\gamma} f_{\text{soft}}(\varphi, \gamma; \boldsymbol{\lambda}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{Y}\varphi - \gamma||_2^2 + \sum_{t=1}^{T} \lambda_t |\gamma_t|$$

Joinly convex in φ and γ - Alternating optimisation

IPOD for Innovative Outliers

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Joinly convex in φ and γ - Alternating optimisation

IPOD

```
while not converged do
```

1. Estimate φ
2. Thresholding $(\boldsymbol{y} - \boldsymbol{Y}\hat{\varphi})$ to obtain optimal γ

```
end while
```

Proximal gradient descent

IPOD

```
while not converged do
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1. Estimate φ
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end while
```

Thresholding

$$r_t = y_t - \varphi y_{t-1} \qquad \Theta_{soft}(r;\lambda) = \begin{cases} r - \lambda & r > \lambda \\ 0 & -\lambda \leq r \leq \lambda \\ r + \lambda & r < \lambda \end{cases}$$

Proximal gradient descent

IPOD

```
while not converged do
```

1. Estimate φ
2. Thresholding $(\boldsymbol{y} - \boldsymbol{Y}\hat{\varphi})$ to obtain optimal γ

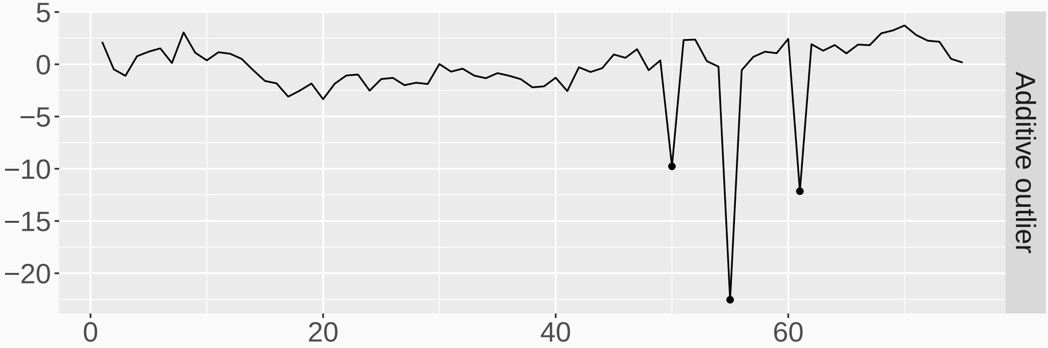
```
end while
```

Same for cross-sectional data

Thresholding

$$r_t = y_t - \varphi y_{t-1} \qquad \Theta_{soft}(r;\lambda) = \begin{cases} r - \lambda & r > \lambda \\ 0 & -\lambda \leq r \leq \lambda \\ r + \lambda & r < \lambda \end{cases}$$

Additive outlier



Additive outlier

$$y_t = \gamma_t + u_t$$

$$u_t = \varphi_1 u_{t-1} + \varphi_2 u_{t-2} + \cdots + \varphi_p u_{t-p} + \varepsilon_t$$

IPOD for Additive Outliers?

Additive outliers

$$y_t = \gamma_t + u_t$$

$$u_t = \varphi_1 u_{t-1} + \varphi_2 u_{t-2} + \cdots + \varphi_p u_{t-p} + \varepsilon_t$$

IPOD Objective function

$$\min_{\gamma} f_{\text{soft}}(\varphi, \gamma; \boldsymbol{\lambda}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{U}\varphi - \gamma||_2^2 + \sum_{t=1}^{T} \lambda_t |\gamma_t|.$$

IPOD for Additive Outliers?

Objective function

$$\min_{\gamma} f_{\text{soft}}(\varphi, \gamma; \lambda) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{U}\varphi - \gamma||_2^2 + \sum_{t=1}^{T} \lambda_t |\gamma_t|$$

IPOD?

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while not converged do
```

1. Estimate φ
2. Thresholding $(\boldsymbol{y} - \boldsymbol{U}\hat{\varphi})$ to obtain optimal γ

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end while
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Proximal gradient descent?

IPOD?

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while not converged do
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1. Estimate φ
2. Thresholding $(\boldsymbol{y} - \boldsymbol{U}\hat{\varphi})$ to obtain optimal γ

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end while
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Thresholding

$$r = y - \hat{u} \qquad \Theta_{soft}(r; \lambda) = \begin{cases} r - \lambda & r > \lambda \\ 0 & -\lambda \leq r \leq \lambda \\ r + \lambda & r < \lambda \end{cases}$$

Proximal gradient descent?

IPOD?

```
while not converged do
```

1. Estimate φ
2. Thresholding $(\boldsymbol{y} - \boldsymbol{U}\hat{\varphi})$ to obtain optimal γ

```
end while
```

Thresholding

No longer solves a consistent objective function!

$$r = y - \hat{u} \qquad \Theta_{soft}(r;\lambda) = \begin{cases} 0 & -\lambda \leq r \leq \lambda \\ r + \lambda & r < \lambda \end{cases}$$

Solving it properly

After some derivation...

For given φ, solving for γ can be expressed as a lasso problem.

$$\min_{\gamma} f_{\text{soft}}(\gamma; \boldsymbol{\lambda}, \varphi) = \frac{1}{2}||\boldsymbol{\Phi}^*\gamma - \boldsymbol{\delta}||_2^2 + \sum_{t=1}^{T} \lambda_t|\gamma_t|$$

```
while not converged do
```

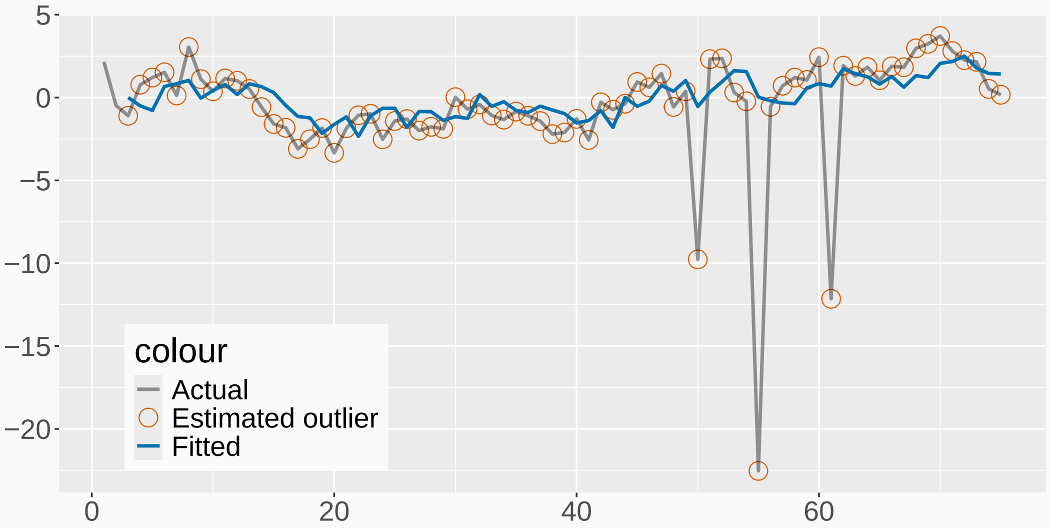
1. Estimate φ
2. Inner proximal gradient descent loop
   - `while not converged do`

     Thresholding to obtain optimal γ

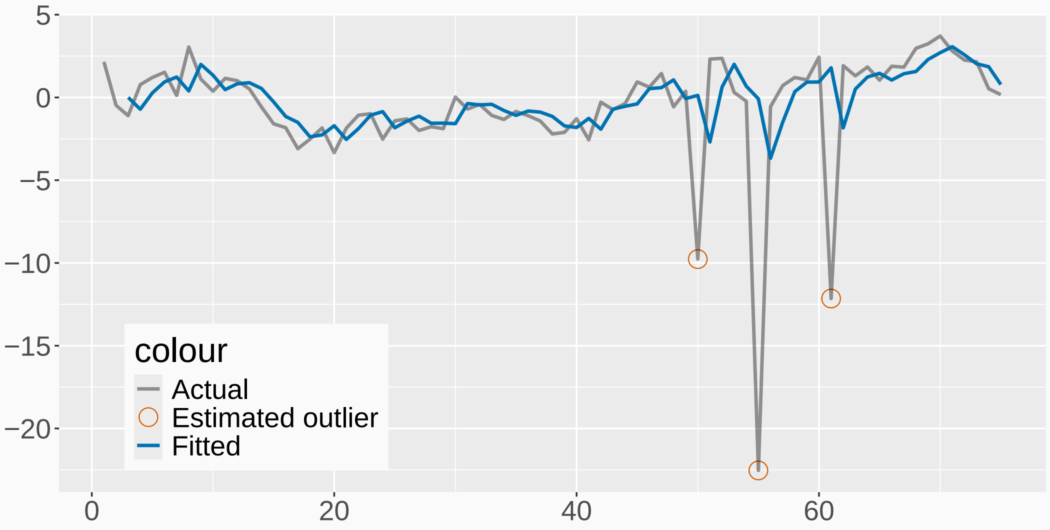
     `end while`

```
end while
```

Optimisation: Iteration 0



Optimisation: Iteration 1



Observation

Outliers can be identified

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Outliers can be identified but **underestimated**.

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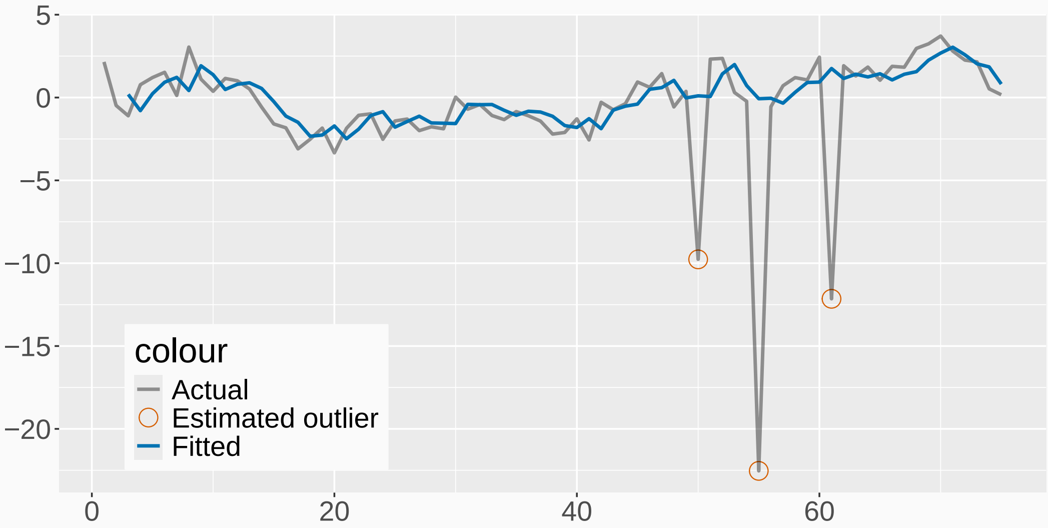
Outliers can be identified but **underestimated**.

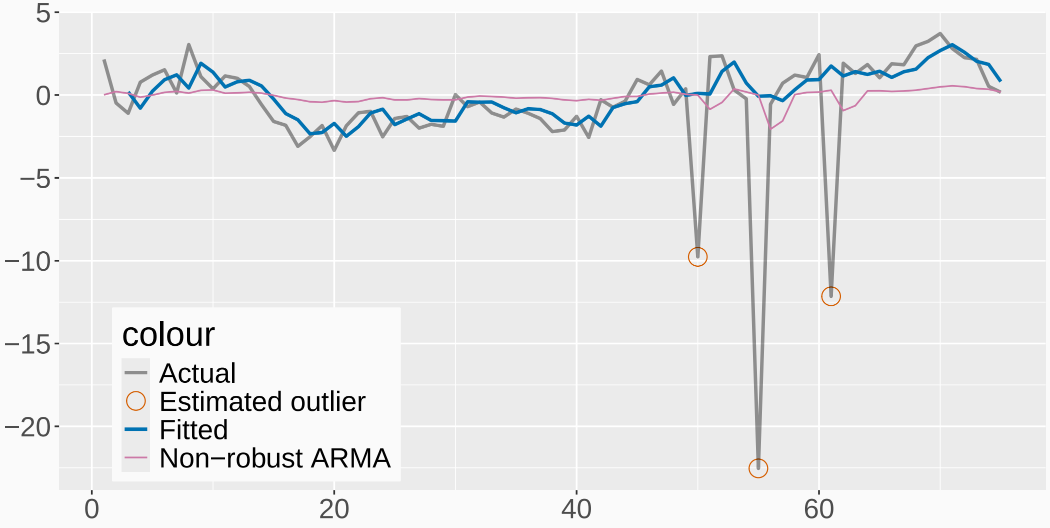
Reweighting

1. Remove the effects of estimated outlier and refit the model
2. Keep $(\boldsymbol{y} - \boldsymbol{U}\hat{\varphi})$ at the locations of the identified outlier as the new estimated outliers
3. Refit the model with the effects of the new outliers removed

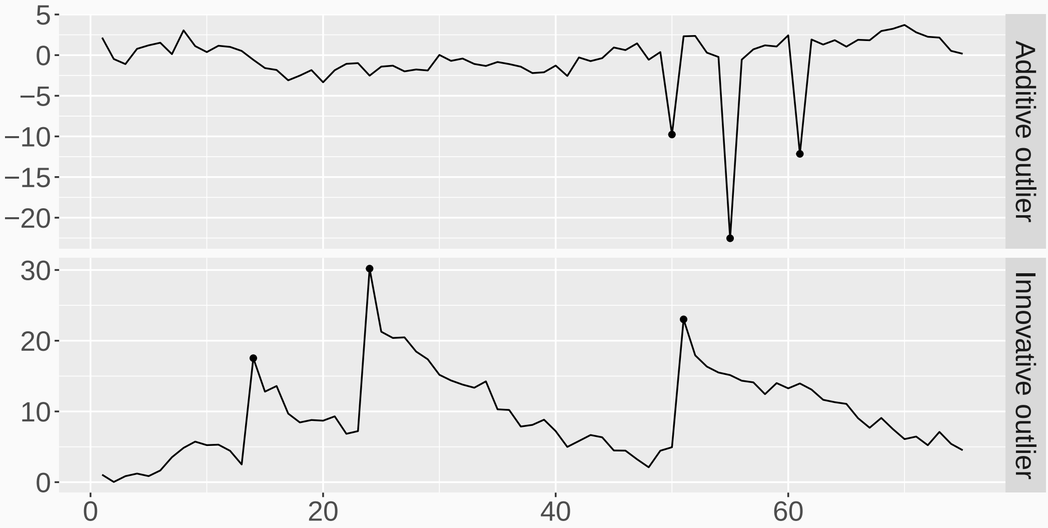
Iteration… and Reweight



Non-robust ARMA



Outliers in time series



AR(1): First obs is an outlier

Objective function

$$\min \sum_{t=2}^{T} \tilde{e}_t^2$$

$$\tilde{e}_2 = y_2 - \gamma_2 - \varphi(y_1 - \gamma_1) = y_2 - \varphi(y_1 - \gamma_1) = e_2 + \varphi\gamma_1$$

$$\tilde{e}_3 = y_3 - \gamma_3 - \varphi(y_2 - \gamma_2) = y_3 - \varphi y_2 = e_3$$

$$\tilde{e}_t = e_t, \text{ where } e_t = y_t - \varphi y_{t-1}$$

Optimal γ

$$\gamma_1^* = \frac{e_2}{\varphi}$$

AR(1): First two obs are outliers

Objective function

$$\min \sum_{t=2}^{T} \tilde{e}_t^2$$

$$\tilde{e}_2 = y_2 - \gamma_2 - \varphi(y_1 - \gamma_1) = y_2 - \gamma_2 - \varphi(e_1 - \gamma_1) = e_2 + \varphi\gamma_1 - \gamma_2$$

$$\tilde{e}_3 = y_3 - \gamma_3 - \varphi(y_2 - \gamma_2) = y_3 - \varphi(y_2 - \gamma_2) = e_3 + \varphi\gamma_2$$

$$\tilde{e}_t = e_t$$

Optimal γ

$$\gamma_2^* = -\frac{e_3}{\varphi} = -\frac{y_3 - \varphi y_2}{\varphi} \qquad \gamma_1^* = -\frac{e_2 - \gamma_2^*}{\varphi} = -\frac{e_2}{\varphi} - \frac{e_3}{\varphi^2}$$

AR(1): First two obs are outliers

Objective function

$$\min \sum_{t=2}^{T} \tilde{e}_t^2$$

$$\tilde{e}_2 = y_2 - \gamma_2 - \varphi(y_1 - \gamma_1) = y_2 - \gamma_2 - \varphi(e_1 - \gamma_1) = e_2 + \varphi\gamma_1 - \gamma_2$$

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Optimal value of γ_i depends on optimal value of γ_j

Optimal γ

$$\gamma_2^* = -\frac{e_3}{\varphi} = -\frac{y_3 - \varphi y_2}{\varphi} \qquad \gamma_1^* = -\frac{e_2 - \gamma_2^*}{\varphi} = -\frac{e_2}{\varphi} - \frac{e_3}{\varphi^2}$$

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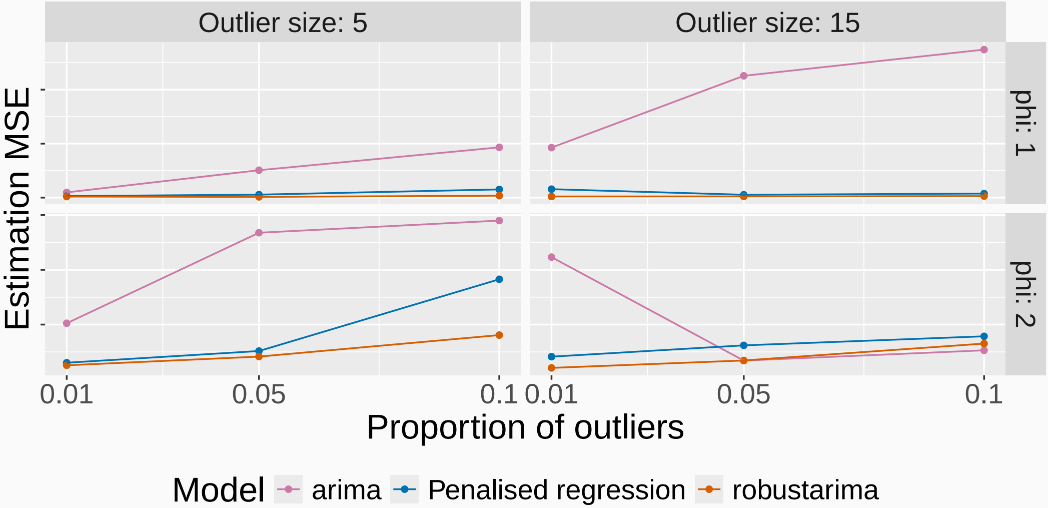
Optimal value of γ_i depends on optimal value of γ_j

Optimal γ

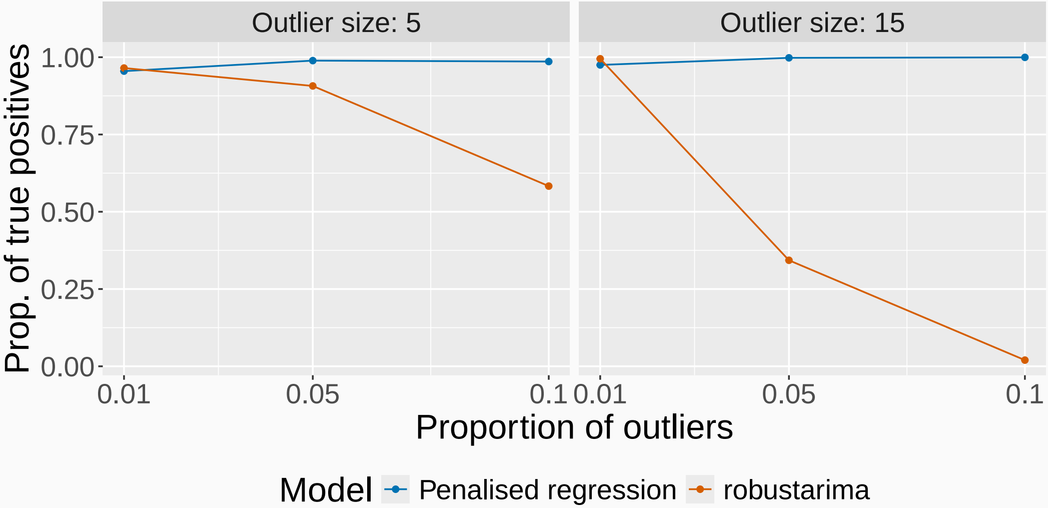
$$\gamma_2^* = -\frac{e_3}{\varphi} = -\frac{y_3}{\varphi} \quad \cdots \quad \frac{\ldots_3}{\varphi^2}$$

Never the case for innovative outliers!

Performance: Model estimation (no reweight)



Performance: Outlier identification



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References

She, Yiyuan, and Art B Owen. 2011. “Outlier detection using nonconvex penalized regression.” *J. Am. Stat. Assoc.* 106 (494): 626–39. https://doi.org/10.1198/jasa.2011.tm10390.