MONASH University

MONASH BUSINESS SCHOOL

Dimension Reduction in Stochastic Optimal Control

Yangzhuoran (Fin) Yang

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Outline

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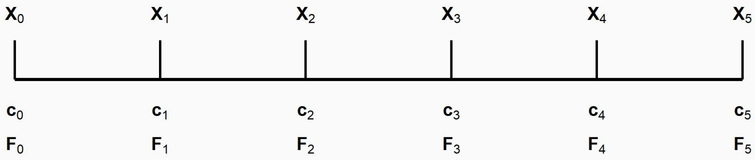
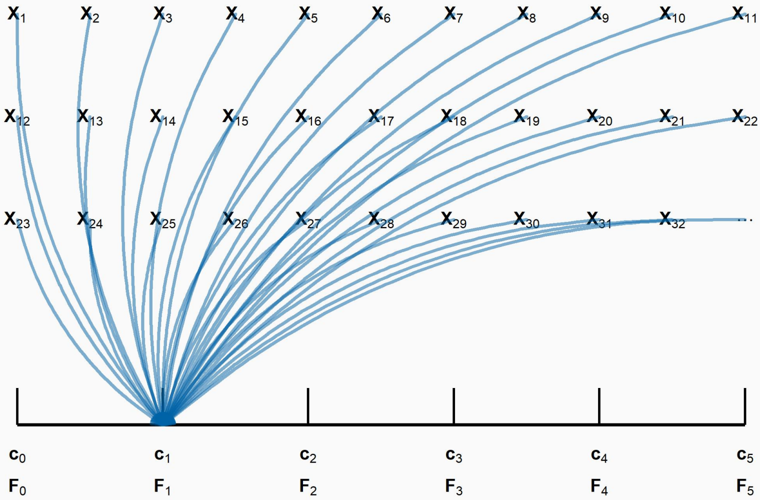
Our setting

In a **finite time horizon**, with available **assets** X_t and **information** F_t (e.g. past returns) at each time point, we try to optimise the **objective** function with a certain utility function with respect to the **control variable** c_t (e.g. consumption, proportion of money invested in each assets).



Our setting


X1 X2 X3 X4 X5 X6 X7 X8 X9 X10 X11
X12 X13 X14 X15 X16 X17 X18 X19 X20 X21 X22
X23 X24 X25 X26 X27 X28 X29 X30 X31 X32 ...
c0 c1 c2 c3 c4 c5
F0 F1 F2 F3 F4 F5

Goals

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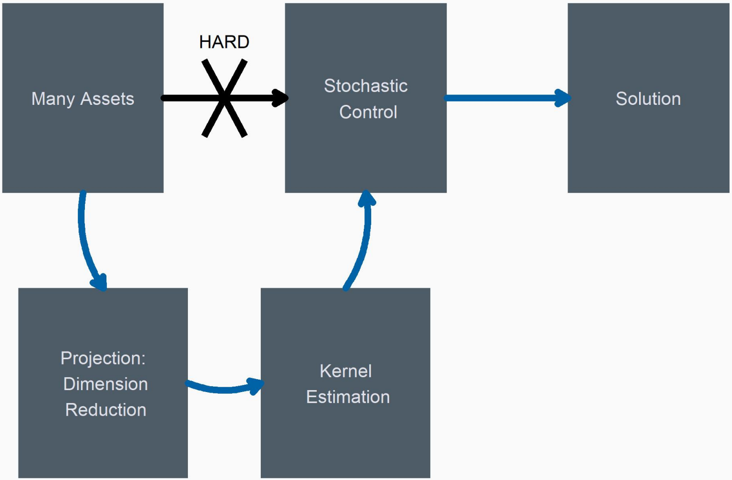
Developing an algorithm that achieves the **optimal portfolio selection** w.r.t. the objective utility function in an **optimal control setting** using **dimension reduction**, where the risky assets are projected onto one risky portfolio using linear regression.

The Big Picture



Significance

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- Overcoming the curse of dimensionality.
- Reducing the required computational power.
- Filling the gap to utilise the dynamics of portfolio selection in the stochastic control theory.

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Significance

- We can do it when others cannot.
- We do it much faster than others.
- We can do it better.

Literature review

Portfolio selection

- Markowitz (1952): Mean-variance analysis
- Fan, Zhang, and Yu (2012): Vast Portfolio Selection With Gross-Exposure Constraints
  - Connection with regression problem

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Portfolio selection

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Optimal control

- Bellman (1957): Bellman equation
  - $V_t(\mathbf{x}_t) = \inf_{\{c_j\}_{j=t}^{T-1}} E\left[\sum_{j=t}^{T-1} U(\mathbf{X_j}, \mathbf{c_j}) + U(\mathbf{X}_T)\right]$
  - $V_t(\mathbf{x}) = \inf_{c_t} \left(U(\mathbf{x}, c_t) + E[V_{t+1}|\mathcal{F}_t]\right)$
  - where V_t is the value function at time t; U is the utility function; $\mathbf{X}_t$ is a vector of states variable at time t; c_t is the control variable at time t

Literature review

- Yao, Li, and Li (2016): Multi-period mean-variance portfolio selection with stochastic interest rate and uncontrollable liability
- Ljungqvist and Sargent (2018): Recursive macroeconomic theory
  - The use of dynamic programming and recursive model in economics
- Hansen and Sargent (2013): Recursive models of dynamic linear economies
  - Discussion the situation where the Bellman equation has a closed form solution

Outline

Projection

The focus of portfolio selection has been on risk minimization.

$$\min_{\omega^T \mathbf{1}=1} Var(\omega^T \mathbf{R}) = \min_{\omega^T \mathbf{1}=1} \omega^T \mathbf{\Sigma} \omega$$

(Fan, Zhang, and Yu 2012)

where **R** is the return vector; **Σ** is its associated covariance matrix; ω is its portfolio allocation vector.

Projection

$$
\begin{aligned}
&\min_{\omega'\mathbf{1}=1} Var(\boldsymbol{\omega}'\mathbf{R}) \\
&= \min_{\boldsymbol{\omega}'\mathbf{1}=1,\xi} E(\boldsymbol{\omega}'\mathbf{R} - \xi)^2 \\
&= \min_{\{\omega_i\}_{i=2}^{q},\xi} E((1 - \sum_{i=2}^{q} \omega_i)R_1 + \omega_2 R_2 + \cdots + \omega_q R_q - \xi)^2 \\
&= \min_{\{\omega_i\}_{i=2}^{q},\xi} E(R_1 - \omega_2(R_1 - R_2) \cdots - \omega_q(R_1 - R_q) - \xi)^2 \\
&= \min_{\{\omega_i\}_{i=2}^{q},\xi} E(R_1 - (\xi + \sum_{i=2}^{q} \omega_i(R_1 - R_i)))^2
\end{aligned}
$$

Projection

$$\min_{\boldsymbol{\omega}'\mathbf{1}=1} Var(\boldsymbol{\omega}'\boldsymbol{R}) = \min_{\{\omega_i\}_{i=2}^{q},\xi} E(R_1 - (\xi + \sum_{i=2}^{q} \omega_i (R_1 - R_i)))^2$$

Solving by OLS

$$R_t^r = \boldsymbol{\omega}_t'\mathbf{R}_t$$

Dimension reduced.

Single Index

Evolution of wealth

$$W_{t+1} = (W_t - \beta_t) \cdot R^f + \beta_t \cdot R_t^r - C_{t+1}$$

- where W_t is the wealth at time t;
- R^f is the accumulation factor for risk free asset from time t to $t+1$;
- R_t^r is the accumulation factor for the risky portfolio from time t to $t+1$;
- β_t is the amount of wealth invested in the risky portfolio at time t;
- C_{t+1} is the consumption made at time $t+1$.

Single Index

Wealth evolution

$$W_{t+1} = (W_t - \beta_t) \cdot R^f + \beta_t \cdot R^r_t - C_{t+1}$$

The value function at time t is

$$f_t(W_t) = \min_{\{C_s\}_{s=t}^{T}, \{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^{T} \delta^{s-t} \cdot (C_s^2 - 2\lambda C_s) + \delta^{T-t} \cdot (W_T^2 - 2\lambda W_T) | \mathcal{F}_t]$$

with terminal condition $f_T(W_T) = W_T^2 - 2\lambda W_T$

We have used a variance-mean utility function.

Single Index

Solving single index

1. Rewrite in the Bellman equation format, we have $V_t(W_t) = \inf_{\mathbf{c}_t, \beta_t} u(W_t, c_t, \beta_t) + \mathbf{E}[V_{t+1}(W_{t+1})]$
2. Mathematical induction:
   a. At time $T - 1$, solve the FOC of the Bellman equation
   b. Assume it is true at time $t + 1$, solve the FOC of the Bellman equation at time t

Single Index

Define

$$\widetilde{W_t} = W_t - \lambda \qquad c_t = C_t - \lambda$$

$$J_t = \begin{bmatrix} R_t^r - R^f \\ -1 \end{bmatrix} \qquad Z_t = \begin{bmatrix} \beta_t \\ c_{t+1} \end{bmatrix} \qquad \mathbb{I}^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Single Index

Define auxiliary variable

HDGF

$$
\begin{cases}
H_t = \mathbb{I}^{22} + D_{t+1} E[J_t J_t'] \\
D_t = \delta (R^f)^2 D_{t+1} \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t]) \\
G_t = (\delta R^f G_{t+1} + \delta R^f (R^f - 2) \lambda D_{t+1}) \\
\qquad \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t]) \\
F_t = (\delta (R^f - 2)^2 \lambda^2 D_{t+1} + 2\delta (R^f - 2) \lambda G_{t+1} + \delta F_{t+1}) \\
\qquad \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t])
\end{cases}
$$

Solution

Theorem

For $t = 0, 1, \cdots, T-1$, the optimal value for the aforementioned problem is

$$f_t(W_t) = \widetilde{W}_t^2 D_t + 2\widetilde{W}_t G_t + F_t - \lambda^2 (\sum_{s=t}^{T-1} \delta^{s+1-t} + \delta^{T-t})$$

and the corresponding optimal strategy is given by

$$Z_t = -H_t^{-1}(D_{t+1}(\widetilde{W}_t R^f + (R^f - 2)\lambda) + G_{t+1})E[J_t]$$

Solution

HDGF

$$
\begin{cases}
H_t = \mathbb{I}^{22} + D_{t+1} E[J_t J_t'] \\
D_t = \delta (R^f)^2 D_{t+1} \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t]) \\
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F_t = (\delta (R^f - 2)^2 \lambda^2 D_{t+1} + 2\delta (R^f - 2) \lambda G_{t+1} + \delta F_{t+1}) \\
\qquad \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t])
\end{cases}
$$

Kernel Estimation

To find the empirical distribution $F_t^n(y; \beta_t)$, we employ the nonparametric kernel estimation.

Kernel estimator

$$\hat{f}_\eta(y) = \frac{1}{\eta n} \sum_{i=1}^{n} \kappa\left(\frac{y - y_i}{\eta}\right)$$

where η is the smoothing parameter bandwidth and the kernel function κ the Gaussian kernel:

$$\kappa(u) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}u^2\right)$$

Upper Bound

As the kernel estimation and the projection approach have been employed, the set of control vector $\{c_i\}_{i=t}^{T-1}$ can be viewed as a sub-optimal to the problem. Given a sub-optimal $\{c_i^*\}_{i=t}^{T-1}$ obtained from the above steps, we can obtain a upper bound of the process

Upper bound

$$\overline{V}_t^*(x_t) = \mathbb{E}[\sum_{j=t}^{T-1} \delta^{j-t} u(X_j, c_j^*) + U(X_T)\delta^{T-t}]$$

Lower bound

We know that

$$\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T \geq \min_{\mathbf{c}_0} \sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T$$

By Jensen's inequality, we can get

$$\inf_{\mathbf{c}_0} \mathbb{E}[\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] \geq$$

$$\mathbb{E}[\min_{\mathbf{c}_0} \sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0]$$

Lower bound

By definition

$$\inf_{\mathbf{c}} \mathbb{E}[\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] = V_0(x_0)$$

Lower bound

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Theorem

We have an lower bound of the value

$$\mathbb{E}[\min_{\mathbf{c}_0} \sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] \leq V_0(x_0)$$

and furthermore when utility functions are linear, we have the equality.

Outline

Competing method: EM

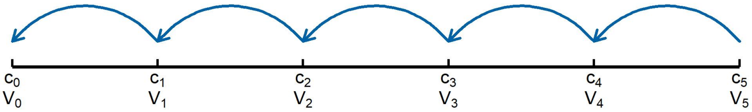
In each iteration

1. Simulate the return of the assets forward in time and calculates the corresponding wealth of the individual using the current parameters
2. Go backward in time to update the parameters of the control variables by optimizing the objective function

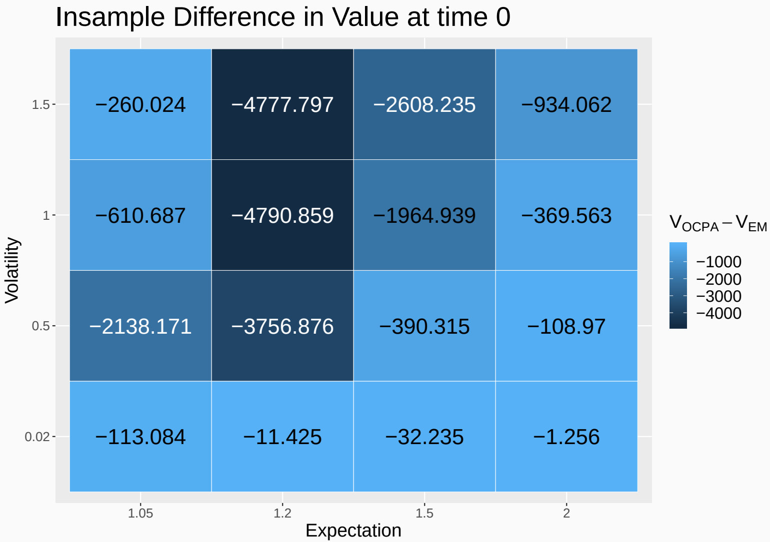
Competing method: EM

E step: Simulate forward in time

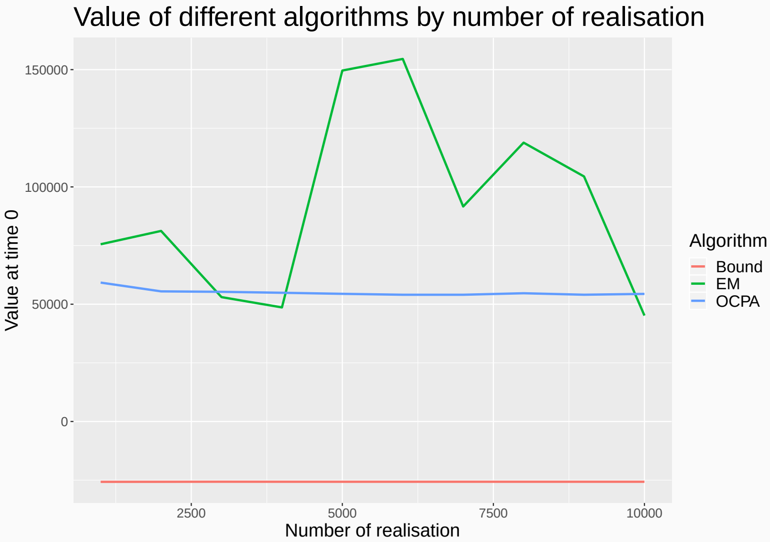
M step: Optimise backward in time



Comparison in Simulation



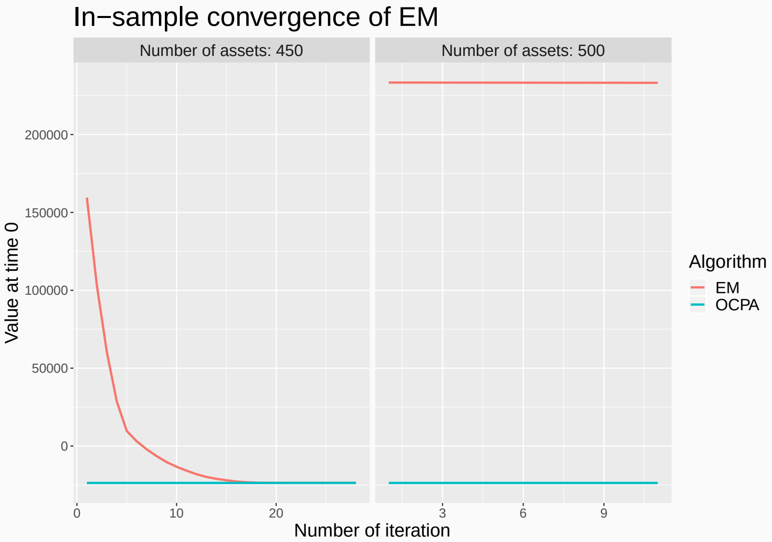
Lower bound



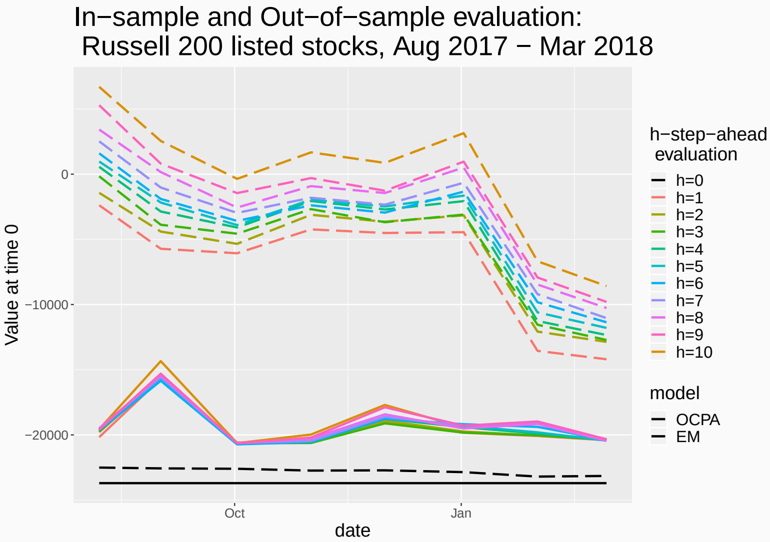
Value of different algorithms by number of realisation
150000
100000
50000
0
Value at time 0
2500
5000
7500
10000
Number of realisation
Algorithm
Bound
EM
OCPA

Time difference

Time difference



Real data and prediction



Outline

Summary

- Theory
  - The solution of the OCPA algorithm
  - Lower bound
- Performance of OCPA
  - Computational advantage
  - Feasibility in high dimension
  - In-sample and out-of-sample performance
  - Robustness towards the different number of realisations

R package stocon

The package is in development. You can install the development version

```
devtools::install_github(FinYang/stocon)
```

The documentation can be found at

https://pkg.yangzhuoranyang.com/stocon

References i

Bellman, RICHARD. 1957. “Dynamic Programming.” *Princeton, USA: Princeton University Press* 1 (2): 3.

Fan, Jianqing, Jingjin Zhang, and Ke Yu. 2012. “Vast Portfolio Selection with Gross-Exposure Constraints.” *Journal of the American Statistical Association* 107 (498): 592–606. https://doi.org/10.1080/01621459.2012.682825.

References ii

Hansen, Lars Peter, and Thomas J Sargent. 2013. *Recursive Models of Dynamic Linear Economies*. Princeton University Press.

Ljungqvist, Lars, and Thomas J Sargent. 2018. *Recursive Macroeconomic Theory*. MIT press.

Markowitz, Harry. 1952. "PORTFOLIO Selection*." *The Journal of Finance* 7 (1): 77–91. `https://doi.org/10.1111/j.1540-6261.1952.tb01525.x`.

References iii

Yao, Haixiang, Zhongfei Li, and Duan Li. 2016. "Multi-Period Mean-Variance Portfolio Selection with Stochastic Interest Rate and Uncontrollable Liability." *European Journal of Operational Research* 252 (3): 837–51.